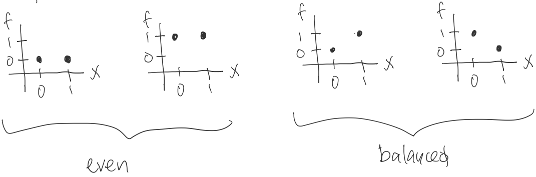
7: Deutsch's Algorithm

Friday, March 26, 2021 3:08 PM

1-Bit Function

x	f(x)
0	
1	

4 possible functions

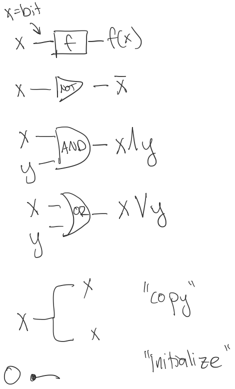


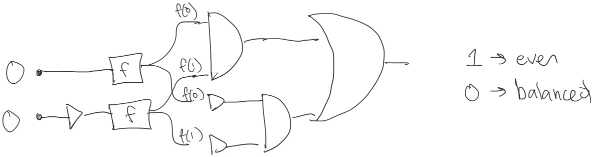
Problem: Given f, decide if f is even or balanced

To design alg: circuit model

Classical:

Gates:


x=bit
x
f
f(x)
x
NOT
$\bar{x}$
x
y
AND
$x \wedge y$
x
y
OR
$x \vee y$
x
x
x
"copy"
"initialize"
0


f(0)
0
0
f
f
f(1)
f(0)
f(1)
1 → even
0 → balanced

Time ——————→

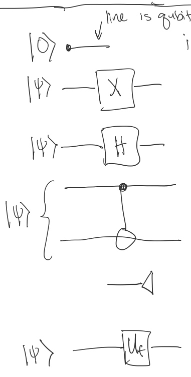
Goal: Use $\boxed{f}$ as few times as possible.

Q What is minimum no. of times $\boxed{f}$ must be used to determine if even or balanced?

A) 0   B) 1   C) 2

"Query Complexity (of Deutsch's Alg)" = # of uses of $\boxed{f}$ needed to solve (problem)

Quantum Gates = Unitaries


line is qubit
|0⟩
|ψ⟩
X
|ψ⟩
H
|ψ⟩
|ψ⟩
U_f

initialize

Paulis

Hadamard

CNOT

Measure in standard basis $\{|0\rangle, |1\rangle\}$

Also

Unitary to implement f ??

Q: Consider if U_f acts on standard basis states in following way:

U_f:

$|0\rangle \longrightarrow |f(0)\rangle$

$|1\rangle \longrightarrow |f(1)\rangle$

ex: $f(0)=1$
$f(1)=0$

$U_f|0\rangle = |1\rangle$
$U_f|1\rangle = |0\rangle$

A. This is a good unitary

B. This is not a good unitary because we haven't defined its action on $|+\rangle$

C. This is not a good unitary because it depends on f.

D. This is not a good unitary because it is not always reversible.

ex: $f(0)=1 \quad f(1)=1$

$U_f|0\rangle = |1\rangle$

$U_f|1\rangle = |1\rangle$

$$\begin{tikzcd} & {|0\rangle} \\ {U_f^\dagger|1\rangle} \arrow[ur] \arrow[dr] & \\ & {|1\rangle} \end{tikzcd}$$

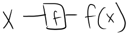
$\Uparrow$

Reverses U_f ?

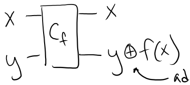
Remove

How do we make f reversible classically

- Classically:



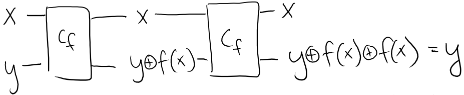
(non reversible)


X
y
C_f
X
$y \oplus f(x)$

(reversible)

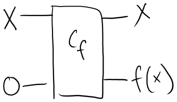
addition mod 2

Self-Inverting

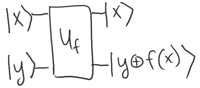

X
y
C_f
X
$y \oplus f(x)$
C_f
X
$y \oplus f(x) \oplus f(x) = y$

- $f(x)=0 \Rightarrow y \oplus 0 \oplus 0 = y$
- $f(x)=1 \Rightarrow y \oplus 1 \oplus 1 = y$

ex:



Reversible Quantum f: just add kets!


$|x\rangle$
$|y\rangle$
U_f
$|x\rangle$
$|y\oplus f(x)\rangle$

★ Must Interpret $|x\rangle, |y\rangle$ as standard basis states ★

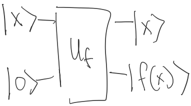
$|\psi\rangle \to$ any state

$|x\rangle, |y\rangle \to$ s.b. states

ex: $f(0)=1$, $f(1)=0$

$$U_f|0\rangle|0\rangle = |0\rangle|0\oplus f(0)\rangle = |0\rangle|1\rangle$$
$$U_f|1\rangle|1\rangle = |1\rangle|1\oplus f(1)\rangle = |1\rangle|1\rangle$$
$$U_f|0\rangle|1\rangle = |0\rangle|1\oplus f(0)\rangle = |0\rangle|0\rangle$$
$$U_f|1\rangle|0\rangle = |1\rangle|0\oplus f(1)\rangle = |1\rangle|0\rangle$$

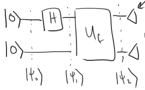
Can check it is reversible (self inverting)


$|x\rangle$
$|0\rangle$
U_f
$|x\rangle$
$|f(x)\rangle$

Once we define how U_f acts on standard basis, we've defined unitary

Ask for ideas: Bomb inspired...

Use Superposition!


$|0\rangle$
$|0\rangle$
H
U_f
$|\psi_0\rangle$
$|\psi_1\rangle$
$|\psi_2\rangle$

measure in standard basis

★ H creates superposition

★ Allows you to ask value of $f(0)$ and $f(1)$ at the same time!

$$|\psi_0\rangle = |00\rangle$$

I+> -> can't apply Uf

$$|\psi_1\rangle = H|0\rangle \otimes I|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(U_f|00\rangle + U_f|10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|0\oplus f(0)\rangle + |1\rangle|0\oplus f(1)\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle|f(0)\rangle + |1\rangle|f(1)\rangle)$$

standard basis states

(If $f(0)=1, f(1)=0$ this is the state $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$)

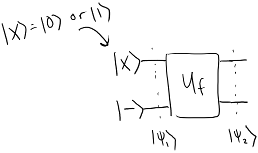
Seems great! We are in superposition of two function values! Classically, if apply C_f once, only get info about one value of function

Now measure in standard basis: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

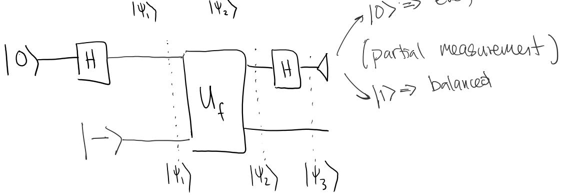
- With prob $\frac{1}{2}$ get outcome $|0\rangle|f(0)\rangle$
- With prob $\frac{1}{2}$ get outcome $|1\rangle|f(1)\rangle$

★ Superposition is not enough to give (usually) a quantum speed-up. Need something more

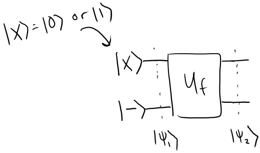
Analyze the following circuits.


$|x\rangle = |0\rangle$ or $|1\rangle$
$|x\rangle$
$|-\rangle$
U_f
$|\psi_1\rangle$
$|\psi_2\rangle$

Explain why $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$


$|0\rangle$
H
$|-\rangle$
U_f
H
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$
$|0\rangle \Rightarrow$ evey
(partial measurement)
$|1\rangle \Rightarrow$ balanced

Analyze the following circuits.


$|x\rangle = |0\rangle$ or $|1\rangle$
$|x\rangle$
$|-\rangle$
U_f
$|\psi_1\rangle$
$|\psi_2\rangle$

Explain why $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|-\rangle$

$$|\psi_2\rangle = U_f|x\rangle|-\rangle = U_f\tfrac{1}{\sqrt{2}}\big(|x\rangle|0\rangle - |x\rangle|1\rangle\big)$$

$$= \tfrac{1}{\sqrt{2}}\big(|x\rangle|f(x)\rangle - |x\rangle|f(x)\oplus 1\rangle\big)$$

$f(x)=0$: $\tfrac{1}{\sqrt{2}}|x\rangle|+\rangle$

$f(x)=1$: $-\tfrac{1}{\sqrt{2}}|x\rangle|+\rangle$

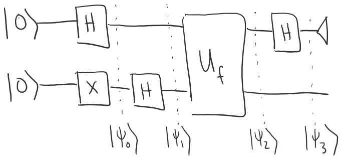
Phase kickback:

Information about $f(x)$ is stored in phase ($+1$ or -1) instead of in $|0\rangle$ or $|1\rangle$ state.

Now know action of U_f on states of form

$$|x\rangle|-\rangle \to (-1)^{f(x)}|x\rangle|-\rangle$$

standard basis


$|0\rangle$
H
H
$|0\rangle$
X
H
U_f
$|\psi_0\rangle$
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$

← Deutsch Algorithm

$$|\psi_1\rangle = |+\rangle|-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle|-\rangle + \tfrac{1}{\sqrt{2}}|1\rangle|-\rangle$$

$$|\psi_2\rangle = U_f \tfrac{1}{\sqrt{2}}|0\rangle|-\rangle + \tfrac{1}{\sqrt{2}}U_f|1\rangle|-\rangle$$

$$= \tfrac{1}{\sqrt{2}}(-1)^{f(0)}|0\rangle|-\rangle + \tfrac{1}{\sqrt{2}}(-1)^{f(1)}|1\rangle|-\rangle$$

↓ Phase Kickback

- If $f(0)=f(1)$

$$= (-1)^{f(0)}\,|+\rangle|-\rangle$$

$$\downarrow H$$

$$|\psi_3\rangle = |0\rangle|-\rangle$$

- If $f(0)\neq f(1)$

$$=(-1)^{f(0)}\,|-\rangle|-\rangle$$

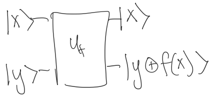
$$\downarrow H$$

$$|\psi_3\rangle = |1\rangle|-\rangle$$

Measure in $\{|0\rangle,|1\rangle\}$ basis. Get $|0\rangle$ if even !
Get $|1\rangle$ if balanced

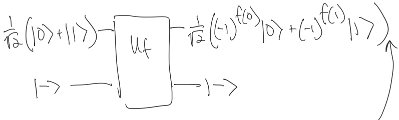
1 Quantum Query!

"Kickback": U_f


$|x\rangle$
$|y\rangle$
U_f
$|x\rangle$
$|y\oplus f(x)\rangle$

← 2nd qubit gets info about $f(x)$

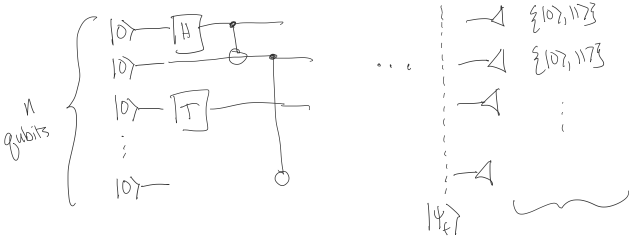
U_f :


$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$
$|-\rangle$
U_f
$\frac{1}{\sqrt{2}}\left((-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle\right)$
$|-\rangle$

1st qubit gets info about $f(x)$...

weird!

More General Circuits


n qubits
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
H
T
. . .
$\{|0\rangle, |1\rangle\}$
$\{|0\rangle, |1\rangle\}$
$|\psi_f\rangle$

$$|0\rangle^{\otimes n} = \underbrace{|0\rangle|0\rangle \cdots |0\rangle}_{n} = |\underbrace{000\cdots0}_{n}\rangle$$

$$\{|00\cdots0\rangle, |000\cdots1\rangle, \cdots\}$$

$$\begin{pmatrix}1\\0\\0\\0\\\vdots\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\\vdots\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \cdots$$

e.g. $|0100110\rangle$

$$|\psi_f\rangle = \sum_{i\in\{0,1\}^n} a_i |i\rangle$$

write as a superposition of standard basis states

When measure, get outcome $|j\rangle$ for $j \in \{0,1\}^n$ with probability

$$|\langle j|\psi_f\rangle|^2 = \left|\langle j| \left(\sum_{i\in\{0,1\}^n} a_i |i\rangle\right)\right|^2$$

$$= \left|\sum_{i\in\{0,1\}^n} a_i \langle j|i\rangle\right|^2 \quad \text{orthonormal}$$

$$= |a_j|^2$$

H, T, CNOT are universal gate set ⇐ Classically, NOT, AND form universal gate set

$$T: \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}$$

⇓ can use to create any unitary*

* to arbitrary small error

* not necessarily efficiently